

# ANALYSIS ON BLIND DECORRELATION OF ISOTROPIC NOISE CORRELATION MATRICES BASED ON SYMMETRIC DECOMPOSITION

Akira Tanaka, Masaaki Miyakoshi\*

Nobutaka Ono

Graduate School of Information Science  
and Technology, Hokkaido University,  
Kita-14, Nishi-9, Kita-ku,  
Sapporo, 060-0814, Japan.

Graduate School of Information Science  
and Technology, The University of Tokyo,  
7-3-1 Hongo Bunkyo-ku,  
Tokyo, 113-8656, Japan.

## ABSTRACT

Recently, a technique named 'blind decorrelation' was proposed by which we can blindly diagonalize correlation matrices of isotropic noises observed by particular crystal-shape sensor arrays. This technique enables us to estimate the power of unknown target signals, which may improve the performance of inverse filters such as the Wiener filter. It was clarified that several classes of crystal-shape arrays achieve the blind decorrelation; and some necessary conditions imposed on a sensor array to realize the blind decorrelation were revealed. However, we do not have a necessary and sufficient condition for a sensor array to achieve the blind decorrelation.

In this paper, we show a necessary and sufficient condition for a sensor array to achieve the blind decorrelation, using a novel matrix analysis scheme named 'symmetric decomposition'.

*Index Terms*— correlation matrices, blind decorrelation, symmetric decomposition, joint diagonalization, inverse filtering

## 1. INTRODUCTION

In the field of array signal processing, such as a multi-channel linear inverse filtering, noise suppression is one of important issues. One of classical and popular linear inverse filters is the minimum variance distortionless response filter (MVDRF), which is defined as the left inverse matrix of an observation matrix minimizing the variance of restored signals. It is well known that the MVDRF is identical to the best linear unbiased estimator (BLUE) when unknown target signals are uncorrelated with observation noises. When the correlation matrix of the unknown target signals (or the power of the target signal in a single channel case) is available, we can use the Wiener filter, which is defined as the minimizer of expected squared errors between the unknown target signals and estimated ones over the signals and the noises. However, the correlation matrix of the unknown target signals is unavailable in practical problems. Thus, in order to adopt the Wiener filter, we have to estimate the correlation matrix of the unknown target signals (or the power of the target signal in a single channel case) in advance. Zelinski proposed a method for estimating the power of the unknown target signal in a single channel case by incorporating the fact that the correlation matrix of the noises is nearly diagonal when the distances between sensors are far enough[1]. However, this assumption for the sensors prevents us from using this method for signals including high-frequency compo-

nents. On the other hand, Ono et al. proved recently that the correlation matrices of isotropic noises observed by particular crystal-shape arrays can be diagonalized blindly[2, 3, 4]. This scheme was named 'blind decorrelation'. The blind decorrelation technique enables us to estimate the power of the unknown target signal when the noises are isotropic even if the distances between sensors are comparatively small since the non-diagonal elements of the correlation matrix of the observations can be noise-free by applying the diagonalizer obtained by the blind decorrelation technique. Moreover, it is reported in [5] that the blind decorrelation technique can be used to estimate the correlation matrix of the noises itself. Ono et al. clarified that several classes of crystal-shape arrays achieve the blind decorrelation[2]; and some necessary conditions imposed on a sensor array to realize the blind decorrelation were revealed[2]. However, we do not have a necessary and sufficient condition for a sensor array to achieve the blind decorrelation.

In this paper, we show a necessary and sufficient condition for a sensor array to achieve the blind decorrelation, using a novel matrix analysis scheme named 'symmetric decomposition'.

## 2. INVERSE FILTERING BY WIENER FILTER

Let  $n$ ,  $m$  ( $m \leq n$ ), and  $t$  be the number of observations, the number of unknown target signals, and the time index (or the frame index in the short time Fourier domain), respectively. Let  $s(t) \in \mathbf{C}^m$ ,  $\mathbf{n}(t) \in \mathbf{C}^n$ , and  $A \in \mathbf{C}^{n \times m}$  be an unknown target signal vector, an observation noise vector, and a given observation matrix consisting of steering vectors of  $s(t)$  (or corresponding to a mixing matrix related with impulse responses between the sources of the target signals and the sensors) with  $\text{rank}(A) = m$ . We assume that the noise vector is zero-mean and uncorrelated with the unknown target signal vector. We assume that an observation vector  $\mathbf{x}(t) \in \mathbf{C}^n$  is given by the following model:

$$\mathbf{x}(t) = A\mathbf{s}(t) + \mathbf{n}(t). \quad (1)$$

Note that we omit the frequency bin index since the following contents does not depend on it. The aim of the inverse filtering is to obtain the signal  $\mathbf{y}(t)$  written as

$$\mathbf{y}(t) = W\mathbf{x}(t), \quad (2)$$

that is as closer to  $\mathbf{s}(t)$  as possible, where the matrix  $W \in \mathbf{C}^{m \times n}$  denotes an inverse filter of  $A$ .

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The Wiener filter (WF) is one of classical and popular inverse filters and is defined as

$$W_{WF} = \operatorname{argmin}_W E_{\mathbf{s}, \mathbf{n}} \|W\mathbf{x}(t) - \mathbf{s}(t)\|^2, \quad (3)$$

where  $E_{\mathbf{s}, \mathbf{n}}$  denotes the mathematical expectation over  $\mathbf{s}(t)$  and  $\mathbf{n}(t)$ . The closed-form solution of the WF is written as

$$W_{WF} = RA^*(ARA^* + Q)^{-1}, \quad (4)$$

where the superscript  $*$  denotes the adjoint (conjugate and transposition) operator; and  $R \in \mathbf{C}^{m \times m}$  and  $Q \in \mathbf{C}^{n \times n}$  denote the correlation matrices of the unknown target signal vector and the observation noise vector defined as

$$\begin{aligned} R &= E_{\mathbf{s}}[\mathbf{s}(t)\mathbf{s}^*(t)], \\ Q &= E_{\mathbf{n}}[\mathbf{n}(t)\mathbf{n}^*(t)], \end{aligned}$$

respectively. Since we assume that the observation noise vector and the unknown target signal vector are uncorrelated, the correlation matrix  $X$  of the observation vector  $\mathbf{x}(t)$  is reduced to

$$\begin{aligned} X &= E_{\mathbf{s}, \mathbf{n}}[\mathbf{x}(t)\mathbf{x}^*(t)] \\ &= E_{\mathbf{s}, \mathbf{n}}[(A\mathbf{s}(t) + \mathbf{n}(t))(A\mathbf{s}(t) + \mathbf{n}(t))^*] \\ &= AE_{\mathbf{s}, \mathbf{n}}[\mathbf{s}(t)\mathbf{s}^*(t)]A^* + E_{\mathbf{s}, \mathbf{n}}[\mathbf{n}(t)\mathbf{n}^*(t)] \\ &= ARA^* + Q. \end{aligned} \quad (5)$$

Substituting Eq.(5) to Eq.(4) yields another expression of the WF written as

$$W_{WF} = RA^*X^{-1}. \quad (6)$$

An estimate of the correlation matrix  $X$  can be easily obtained by the given observations. However, we have to estimate the correlation matrix  $R$  in order to use the WF.

### 3. BLIND DECORRELATION OF CORRELATION MATRICES OF ISOTROPIC NOISES

In case of  $m = 1$ , Zelinski estimated the power of the target signal by the non-diagonal elements of the correlation matrix of the observations in [1], incorporating the fact that the cross power spectrum of the observation noises is nearly equal to zero when the distance between any two sensors are far enough. However, this requirement for the sensors prevents us from adopting this method for target signals including high-frequency components.

On the other hand, Ono et al. clarified that the correlation matrices of isotropic noises observed by particular crystal-shape arrays can be diagonalized by a constant non-singular matrix[2, 3, 4]. This scheme was named 'blind decorrelation'. In this section, we review the theory of the blind decorrelation technique.

We define the isotropic observation noises as the noises satisfying the following two properties:

- 1) the noise power spectrum on each sensor is identical,
- 2) the noise cross spectrum is determined by only a distance between sensors.

The correlation matrix of the isotropic observation noises can be written as

$$Q = \begin{bmatrix} \Gamma(r_{11}) & \cdots & \Gamma(r_{1n}) \\ \vdots & \ddots & \vdots \\ \Gamma(r_{n1}) & \cdots & \Gamma(r_{nn}) \end{bmatrix}, \quad (7)$$

where  $r_{ij}$  denote the distance between the  $i$ -th sensor and the  $j$ -th sensor; and  $\Gamma$  denotes some function that gives the (cross) power

spectrum which only depends on the distance between two sensors. Note that the matrix  $Q$  defined by Eq.(7) is symmetric since  $r_{ij} = r_{ji}$  holds for any  $i, j \in \{1, \dots, n\}$ . For instance, the correlation matrix of the isotropic observation noises observed by a unit square array whose sensors are numbered clockwise, is written as

$$Q = \begin{bmatrix} \Gamma(0) & \Gamma(1) & \Gamma(\sqrt{2}) & \Gamma(1) \\ \Gamma(1) & \Gamma(0) & \Gamma(1) & \Gamma(\sqrt{2}) \\ \Gamma(\sqrt{2}) & \Gamma(1) & \Gamma(0) & \Gamma(1) \\ \Gamma(1) & \Gamma(\sqrt{2}) & \Gamma(1) & \Gamma(0) \end{bmatrix}. \quad (8)$$

The following theorems are the important results for the blind decorrelation shown in [2]

**Theorem 1** [2] *If the correlation matrix  $Q$  of the isotropic observation noises defined by Eq.(7) can be diagonalized by a constant unitary matrix  $U$  for any function  $\Gamma$ , then a set of distances from the  $i$ -th sensor to the others is identical for any  $i \in \{1, \dots, n\}$ .*

**Theorem 2** [2] *If the sensors are positioned to all vertexes of a shape belonging to one of the following five classes of shapes:*

- 1) regular polygons,
- 2) rectangular,
- 3) regular polygonal prisms,
- 4) rectangular solid,
- 5) regular polyhedrons,

*then the correlation matrix  $Q$  of the isotropic observation noises observed by the array can be diagonalized by a constant unitary matrix  $U$ .*

Please refer to [2] for more details of the theoretical results including the closed-form of the unitary matrix  $U$  for each class.

Once a constant diagonalizer  $U$  is given, the correlation matrix of the observations can be transformed to

$$U^*XU = U^*ARA^*U + \Lambda, \quad (9)$$

where  $\Lambda$  denotes a diagonal matrix satisfying  $Q = U\Lambda U^*$ . Thus, the non-diagonal elements of  $U^*XU$  are noise-free, which means that we can estimate the powers of the unknown target signals[3, 4]. Moreover, it is reported in [5] that the blind decorrelation technique can be used to estimate the correlation matrix of the isotropic observation noises itself.

As shown above, we have a necessary condition and a sufficient condition for the blind decorrelation. However, we do not have a necessary and sufficient condition for a sensor array to achieve the blind decorrelation.

### 4. ANALYSES FOR BLIND DECORRELATION BASED ON SYMMETRIC DECOMPOSITION

In this section, we give a necessary and sufficient condition for a sensor array to achieve the blind decorrelation.

#### 4.1. Mathematical Preliminaries

Let  $\mathcal{D} = \{r_1, \dots, r_K\}$  be the set of distances between sensors, where  $K$  denotes the number of different distances. Then, the correlation matrix  $Q$  of the observation noises can be decomposed as

$$Q = \sum_{i=1}^K \Gamma(r_i) B_i, \quad (10)$$

called 'symmetric decomposition' of  $Q$ . The set of matrices  $\mathcal{B} = \{B_1, \dots, B_K\}$  is called the basis of the symmetric decomposition of  $Q$ . Note that each  $B_i$  is a real symmetric matrix. For instance, the symmetric decomposition of the example Eq.(8) is given as follows:

$$Q = \Gamma(0)B_1 + \Gamma(1)B_2 + \Gamma(\sqrt{2})B_3, \quad (11)$$

where

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Note that the basis of the symmetric decomposition of  $Q$  always includes the identity matrix which is written as  $B_1$  hereafter. Here we introduce important theorems concerned with a joint diagonalization of Hermitian matrices.

**Theorem 3** [6] *Let  $B_i \in \mathbf{C}^{n \times n}$  ( $i \in \{1, \dots, K\}$ ) be Hermitian matrices. There exist a unitary matrix  $U$ , such that  $U^* B_i U$  is diagonal for any  $i \in \{1, \dots, K\}$ , if and only if*

$$B_i B_j = B_j B_i \quad (12)$$

holds for any  $i, j \in \{1, \dots, K\}$ .

**Theorem 4** [6] *Let  $B_i \in \mathbf{C}^{n \times n}$  ( $i \in \{1, \dots, K\}$ ) be Hermitian matrices satisfying*

$$\mathcal{R}(B_i) \subset \mathcal{R}(B_1)$$

for any  $i \in \{1, \dots, K\}$ , where  $\mathcal{R}(X)$  denotes the linear subspace spanned by the columns of the matrix  $X$ .

*There exists a non-singular matrix  $T$  such that  $T^* A_i T$  is diagonal for any  $i \in \{1, \dots, K\}$ , if and only if*

1.  $B_i B_1^-$  is semisimple and its eigenvalues are real for any  $i \in \{1, \dots, K\}$ ,
2.  $B_i B_1^- B_j = B_j B_1^- B_i$  holds for any  $i, j \in \{1, \dots, K\}$ ,

for a certain  $B_1^-$  (a generalized inverse matrix of  $B_1$ ).

## 4.2. Main Result

Let  $\mathcal{Q}$  be the set of the matrices defined by Eq.(7) for any  $\Gamma$  and let  $\mathcal{Q}_1$  be the subset of  $\mathcal{Q}$  including all non-negative matrices defined by Eq.(7). The following theorem is the main result of this paper.

**Theorem 5** *The following four statements are equivalent each other:*

- 1) *There exist a non-singular matrix  $T$  such that  $T^* Q T$  is diagonal for any  $Q \in \mathcal{Q}$ .*
- 2) *There exist a non-singular matrix  $T$  such that  $T^* Q T$  is diagonal for any  $Q \in \mathcal{Q}_1$ .*
- 3) *There exist a non-singular matrix  $T$  such that  $T^* B_i T$  is diagonal for any  $B_i \in \mathcal{B}$ .*
- 4)  *$B_i B_j = B_j B_i$  holds for any  $i, j \in \{1, \dots, K\}$ .*

## Proof

1)  $\rightarrow$  2)

It is trivial since  $\mathcal{Q}_1 \subset \mathcal{Q}$ .

2)  $\rightarrow$  3)

Let  $T$  be a non-singular matrix such that  $T^* Q T$  is diagonal for any  $Q \in \mathcal{Q}_1$  and let  $\beta_i$ , ( $i = 2, \dots, K$ ) be an arbitrary number smaller than the minimum eigenvalue of  $B_i$ . Then the set of matrices written as

$$\mathcal{C} = \{C_1, C_2, \dots, C_K\},$$

is a subset of  $\mathcal{Q}_1$ , where  $C_1 = B_1 = I$  and  $C_i = B_i - \beta_i I$  for any  $i \in \{2, \dots, K\}$ . Thus  $T^* C_i T$  is diagonal for any  $i \in \{1, \dots, K\}$ . Since any linear combinations of the elements in  $\mathcal{C}$  can be also diagonalized by  $T$ , any elements in  $\mathcal{B}$  can be diagonalized by  $T$ .

3)  $\rightarrow$  1)

Let  $T$  be a non-singular matrix such that  $T^* B_i T$  is diagonal for any  $B_i \in \mathcal{B}$ . Since any linear combinations of the elements in  $\mathcal{B}$  can be also diagonalized by  $T$ ,  $T^* Q T$  is also diagonal for any  $Q \in \mathcal{Q}$ .

3)  $\rightarrow$  4)

Since any  $B_i \in \mathcal{B}$  can be jointly diagonalized by  $T$  and  $\mathcal{R}(B_i) \subset \mathcal{R}(B_1) = \mathcal{R}(I)$ ,

$$B_i B_j = B_j B_i$$

is obtained for any  $i, j \in \{1, \dots, K\}$  by substituting  $B_1^- = I$  to the second condition in Theorem 4.

4)  $\rightarrow$  3)

According to Theorem 3, there exists a unitary matrix  $U$  such that  $U^* B_i U$  is diagonal for any  $B_i \in \mathcal{B}$  and it is trivial that the unitary matrix  $U$  is non-singular. ■

The most remarkable knowledge obtained by Theorem 5 is the equivalency between 2) and 4). Given a class of the correlation matrix Eq.(7), we can determine whether it can be diagonalized by a constant non-singular matrix independent from the function  $\Gamma$  or not, by testing the commutativity of all pairs of the matrices in the basis of the symmetric decomposition. Thus, we can enumerate all layouts of a sensor array with a finite number of sensors that achieves the blind decorrelation. However, the test for all combinations of the basis requires large amounts of computational costs especially in case of large  $n$ . In fact, we have no layout which is not included in the arrays specified by Theorem 2 at the present time.

It is also clarified by the statement 4) of Theorem 5 that when we have a class of matrix Eq.(7) that can be diagonalized by a constant non-singular matrix, the non-singular matrix that diagonalizes Eq.(7) can be unitary, which means that the class of matrix defined by Eq.(7) has invariant eigenvectors.

## 4.3. Algorithm for Obtaining a Joint Diagonalizer

We also introduce an algorithm for obtaining a joint diagonalizer of all elements in  $\mathcal{B}$  which is basically along with [7].

Let  $\mathcal{B} = \{B_1, \dots, B_K\}$  be the basis of the symmetric decomposition of  $Q$  whose elements can be jointly diagonalized. We ignore the matrix  $B_1$  since  $B_1 = I$  can be diagonalized by an arbitrary unitary matrix. Let

$$B_2 = P_2 \Lambda_2 P_2^* \quad (13)$$

be the eigenvalue decomposition of  $B_2$ , then

$$P_2^* B_2 P_2 = \Lambda_2 \quad (14)$$

is trivially diagonal. If  $B_2$  does not have repeated eigenvalues,  $P_2$  is uniquely determined except the ambiguity of permutation of columns. Thus,  $P_2$  jointly diagonalizes  $B_i$  for any  $i \in \{2, \dots, K\}$ .

On the other hand, if  $B_2$  has repeated eigenvalues,  $\Lambda_2$  is reduced to the form of

$$\Lambda_2 = \begin{bmatrix} \lambda_1^{(2)} I_{r_1^{(2)}} & & O \\ & \ddots & \\ O & & \lambda_{N_2}^{(2)} I_{r_{N_2}^{(2)}} \end{bmatrix}, \quad (15)$$

where  $I_r$  denotes the identity matrix of degree  $r$ ;  $N_2$  denotes the number of different eigenvalues of  $B_2$ ; and  $r_k^{(2)}$  denotes the multiplicity of the eigenvalue  $\lambda_k^{(2)}$  of  $B_2$  satisfying

$$r_1^{(2)} + \dots + r_{N_2}^{(2)} = n. \quad (16)$$

Then,  $D_3 = P_2^* B_3 P_2$  reduced to a block-diagonal matrix

$$D_3 = \begin{bmatrix} H_1^{(3)} & & O \\ & \ddots & \\ O & & H_{N_2}^{(3)} \end{bmatrix}, \quad (17)$$

where  $H_k^{(3)}$  denotes a Hermitian matrix of degree  $r_k^{(2)}$ .

Let

$$H_k^{(3)} = P_k^{(3)} \Lambda_k^{(3)} P_k^{(3)*} \quad (18)$$

be the eigenvalue decomposition of  $H_k^{(3)}$  and let  $P_3$  be the block-diagonal unitary matrix written as

$$P_3 = \begin{bmatrix} P_1^{(3)} & & O \\ & \ddots & \\ O & & P_{N_2}^{(3)} \end{bmatrix}, \quad (19)$$

then, then

$$P_3^* P_2^* B_3 P_2 P_3 = P_3^* D_3 P_3 = \Lambda_3$$

must be diagonal, where

$$\Lambda_3 = \begin{bmatrix} \Lambda_1^{(3)} & & O \\ & \ddots & \\ O & & \Lambda_{N_2}^{(3)} \end{bmatrix}. \quad (20)$$

Since from Eq.(15),

$$P_3^* P_2^* B_2 P_2 P_3 = P_3^* \Lambda_2 P_3 = \Lambda_2$$

is also diagonal trivially,  $(P_2 P_3)$  is a joint diagonalizer of  $B_2$  and  $B_3$ .

If  $H_k^{(3)}$  does not have repeated eigenvalues,  $P_3$  is uniquely determined, which implies that  $(P_2 P_3)$  jointly diagonalizes  $B_i$  for any  $i \in \{2, \dots, K\}$ .

On the other hand, if  $H_k^{(3)}$  has repeated eigenvalues, we have a joint diagonalizer of  $B_2$ ,  $B_3$  and  $B_4$  by similar operations for  $(P_2 P_3)^* B_4 (P_2 P_3)$ . Finally, we have a joint diagonalizer of all elements in  $\mathcal{B}$  by applying the same operations to  $B_k$  successively.

For instance, we give a joint diagonalizer for the example Eq.(8) along with the above algorithm. The eigenvalue decomposition of  $B_2$  defined by Eq.(11) is given as

$$B_2 = P_2 \Lambda_2 P_2^*, \quad (21)$$

where

$$P_2 = \begin{bmatrix} 1/2 & 1/2 & 0 & 1/\sqrt{2} \\ 1/2 & -1/2 & -1/\sqrt{2} & 0 \\ 1/2 & 1/2 & 0 & -1/\sqrt{2} \\ 1/2 & -1/2 & 1/\sqrt{2} & 0 \end{bmatrix},$$

and

$$\Lambda_2 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and  $D_3 = P_2^* B_3 P_2$  is reduced to

$$D_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

which is already diagonal. Thus, in this case,  $P_2$  gives a joint diagonalizer of  $B_2$  and  $B_3$ .

From this example and the fact that the matrix consisting of the eigenvectors of a real symmetric matrix can be a real orthogonal one, we can choose a real orthogonal matrix as a joint diagonalizer, while a complex unitary diagonalizer is given in [2].

## 5. CONCLUSION

In this paper, we clarified a necessary and sufficient condition for a sensor array to achieve the blind decorrelation for isotropic noises, using a novel matrix analysis scheme named symmetric decomposition of a matrix. Enumerating all layouts of a sensor array that achieves the blind decorrelation on the basis of our theorem is one of future works that should be resolved.

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