

# R-MEANS LOCALIZATION: A SIMPLE ITERATIVE ALGORITHM FOR RANGE-DIFFERENCE-BASED SOURCE LOCALIZATION

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## ABSTRACT

In this paper, we present a simple iterative algorithm for range-difference (RD) based localization. The RD-based localization is a kind of nonlinear optimization problem and generally it has no closed-form solution. Through auxiliary function approach, we derive iterative update rules without any tuning parameters, which just consists of 1) averaging source-sensor distances, 2) averaging the source positions estimated by updating source-sensor distance on each sensor with the source-direction fixed. Due to the resemblance of the iterative averaging to  $k$ -means clustering, we call it *r-means localization*. The convergence of the algorithm is guaranteed. The acceleration of the convergence is also investigated.

**Index Terms**— source localization, range difference, auxiliary function,  $k$ -means clustering, acceleration

## 1. INTRODUCTION

Range-difference (RD) based source localization is one of the most fundamental localization methods and it has been applied to various applications such as surveillance, navigation, wireless communications, teleconferencing, and so on. The RD information generally includes observation noise and minimizing the squared errors is a rational criterion. However, due to the nonlinear form of the objective function, the problem has no closed-form solutions.

One research stream is aiming to find a closed-form solution based on different criterion [1, 2, 3], or specific sensor configuration [4]. Generally, closed-form solutions are very useful due to the simple and light calculation. While, another approach is iterative algorithm for directly solving the least square problem, which includes the steepest descent method and Newton's method [5]. Although iterative algorithm yields more accurate estimation, it suffers from cumbersome calculation such as tuning step size, selecting initial values, and exceptional handling for non-converged cases.

Recently, for deriving a simple and effective iterative algorithm, auxiliary function approach has been investigated in signal processing field. It can be considered to be an extension of EM algorithm. The famous example is nonnegative matrix factorization [6] and it has been applied to solve other optimization problems [7, 8, 9].

In the framework of distributed microphone array, we have also investigated to apply the auxiliary function approach to this kind of problem [10]. In this paper, we present a simple

iterative algorithm for RD-based localization, which consists of simple iterative averaging operations. Due to the resemblance to  $k$ -means clustering, we call it *r-means localization*. The convergence of the algorithm is guaranteed and the acceleration of the convergence is also considered. The fundamental validity is shown by simulative experiments.

## 2. FORMULATION OF RD-BASED LOCALIZATION

Let  $\mathbf{s} = (x \ y \ z)^t$  and  $\mathbf{r}_m = (u_m \ v_m \ w_m)^t$  ( $1 \leq m \leq L$ ) denote positions of a single source and sensors, respectively, where  $^t$  denotes vector transpose. Suppose that the sensor positions are known and range differences:

$$d_{mn} = |\mathbf{s} - \mathbf{r}_m| - |\mathbf{s} - \mathbf{r}_n| + \varepsilon_{mn} \quad (1)$$

are observed where  $\varepsilon_{mn}$  denotes observation noise. Here, let's consider the problem estimating source position  $\mathbf{s}$  by minimizing the least square:

$$J(\mathbf{s}) = \sum_{m=1}^L \sum_{n=1}^L (|\mathbf{s} - \mathbf{r}_m| - |\mathbf{s} - \mathbf{r}_n| - d_{mn})^2. \quad (2)$$

## 3. DERIVATION OF R-MEANS LOCALIZATION

### 3.1. Auxiliary function approach

Because eq. (2) includes square roots of position coordinates  $x, y, z$ , minimizing eq. (2) is a nonlinear optimization problem and there are no closed-form solutions. In order to derive an effective iterative algorithm, applying the auxiliary function approach is considered here.

Generally, in the auxiliary function approach, in order to find parameters  $\Theta$  to minimize an objective function  $J(\Theta)$ , an auxiliary function  $Q(\Theta, \bar{\Theta})$  to satisfy

$$J(\Theta) = \min_{\Theta} Q(\Theta, \bar{\Theta}) \quad (3)$$

is utilized, where  $\bar{\Theta}$  is called auxiliary variables. The principle of the auxiliary function method is based on the fact that  $J(\Theta)$  is non-increasing under the updates:

$$\bar{\Theta}^{(k+1)} = \operatorname{argmin}_{\bar{\Theta}} Q(\Theta^{(k)}, \bar{\Theta}), \quad (4)$$

$$\Theta^{(k+1)} = \operatorname{argmin}_{\Theta} Q(\Theta, \bar{\Theta}^{(k+1)}), \quad (5)$$

where  $k$  is the index of iterations. The brief proof is given by the following.

1.  $Q(\Theta^{(k)}, \bar{\Theta}^{(k+1)}) = J(\Theta^{(k)})$  from eq. (3) and eq. (4)
2.  $Q(\Theta^{(k+1)}, \bar{\Theta}^{(k+1)}) \leq Q(\Theta^{(k)}, \bar{\Theta}^{(k+1)})$  from eq. (5)
3.  $J(\Theta^{(k+1)}) \leq Q(\Theta^{(k+1)}, \bar{\Theta}^{(k+1)})$  from eq. (3)

then,

$$J(\Theta^{(k+1)}) \leq J(\Theta^{(k)}), \quad (6)$$

which guarantees non-increasing of objective functions. Thus, the sequence  $J(\Theta^{(k)})$  converges when  $J(\Theta)$  is bounded below. For efficient updates, it is necessary that eq. (4) and eq. (5) are given in closed forms.

### 3.2. Derivation of auxiliary function

There does not always exist an auxiliary function for any optimization problems such that it satisfies eq. (3) and has closed form solutions for eq. (4) and eq. (5). Fortunately, we can find a useful auxiliary function in this problem.

For simplicity, let

$$r_m = |\mathbf{s} - \mathbf{r}_m|, \quad (7)$$

$$\bar{d}_m = \frac{1}{L} \sum_{n=1}^L d_{mn}. \quad (8)$$

First, two lemmas are shown.

**Lemma 1** For any  $\bar{r}$ ,

$$\begin{aligned} & \frac{1}{2L} \sum_{m=1}^L \sum_{n=1}^L (r_m - r_n - d_{mn})^2 \\ & \leq \sum_{m=1}^L (r_m - (\bar{r} + \bar{d}_m))^2 + C, \end{aligned} \quad (9)$$

holds where

$$C = \sum_{m=1}^L (\bar{r} + \bar{d}_m)^2 + \frac{1}{2L} \sum_{m=1}^L \sum_{n=1}^L d_{mn}^2 + L\bar{r}^2. \quad (10)$$

The equality of eq. (9) is valid if and only if

$$\bar{r} = \frac{1}{L} \sum_{m=1}^L r_m. \quad (11)$$

**Proof:** Calculating the left side of eq. (9), we have

$$\begin{aligned} & \frac{1}{2L} \sum_{m=1}^L \sum_{n=1}^L (r_m - r_n - d_{mn})^2 \\ & = \sum_{m=1}^L r_m^2 + \frac{1}{2L} \sum_{m=1}^L \sum_{n=1}^L d_{mn}^2 \\ & \quad - \frac{1}{L} \sum_{m=1}^L \sum_{n=1}^L r_m r_n - 2 \sum_{m=1}^L \bar{d}_m r_m. \end{aligned} \quad (12)$$

Here, note that

$$\left( \frac{1}{L} \sum_m r_m - \bar{r} \right)^2 = \frac{1}{L^2} \sum_{m,n} r_m r_n - \frac{2}{L} \bar{r} \sum_m r_m + \bar{r}^2 \geq 0, \quad (13)$$

that is,

$$-2\bar{r} \sum_m r_m + L\bar{r}^2 \geq -\frac{1}{L} \sum_{m,n} r_m r_n \quad (14)$$

holds for any  $\bar{r}$ . Applying eq. (14) to eq. (12) and eliminating the cross terms of  $r_m$  and  $r_n$ , we have

$$\begin{aligned} & \frac{1}{2L} \sum_{m=1}^L \sum_{n=1}^L (r_m - r_n - d_{mn})^2 \\ & \leq \sum_{m=1}^L r_m^2 + \frac{1}{2L} \sum_{m=1}^L \sum_{n=1}^L d_{mn}^2 \\ & \quad - 2\bar{r} \sum_{m=1}^L r_m + L\bar{r}^2 - 2 \sum_{m=1}^L \bar{d}_m r_m, \\ & = \sum_{m=1}^L (r_m - (\bar{r} + \bar{d}_m))^2 + C. \end{aligned} \quad (15)$$

From eq. (13), it is clear that the equality is valid obviously if and only if  $\bar{r} = (1/L) \sum_{m=1}^L r_m$ . ■

**Lemma 2** For any vector  $\mathbf{x}$ , any unit vector  $\mathbf{e}$ , and positive scalar  $a$ ,

$$(|\mathbf{x}| - a)^2 \leq |\mathbf{x} - a\mathbf{e}|^2 \quad (16)$$

holds. The equality is valid if and only if  $\mathbf{e} = \mathbf{x}/|\mathbf{x}|$ .

**Proof:**

$$\begin{aligned} & |\mathbf{x} - a\mathbf{e}|^2 - (|\mathbf{x}| - a)^2 \\ & = |\mathbf{x}|^2 - 2a\mathbf{x} \cdot \mathbf{e} + a^2|\mathbf{e}|^2 - |\mathbf{x}|^2 + 2a|\mathbf{x}| - a^2 \\ & = 2a(|\mathbf{x}| - \mathbf{x} \cdot \mathbf{e}) \\ & = 2a|\mathbf{x}|(1 - \cos \theta) \geq 0, \end{aligned} \quad (17)$$

where  $\theta$  is the angle between  $\mathbf{x}$  and  $\mathbf{e}$ . The equality is valid if and only if  $\cos \theta = 1$ , which means  $\mathbf{e} = \mathbf{x}/|\mathbf{x}|$ . ■

From Lemma 1 and Lemma 2, an auxiliary function is obtained as shown in the following lemma.

**Lemma 3** When the condition:

$$\bar{r} + \bar{d}_m \geq 0 \quad (18)$$

is satisfied for any  $m$ , an auxiliary function for eq. (2) is obtained by

$$Q(\mathbf{s}, \bar{\Theta}) = 2L \sum_{m=1}^L (\mathbf{s} - (\mathbf{r}_m + (\bar{r} + \bar{d}_m)\mathbf{e}_m))^2 + 2LC, \quad (19)$$

where  $\bar{\Theta} = \{\bar{r}, \mathbf{e}_m | 1 \leq m \leq L\}$ .

**Proof:** From Lemma 1 and letting  $\mathbf{x} = \mathbf{s} - \mathbf{r}_m$  and  $a = \bar{r} + \bar{d}_m$  in Lemma 2, it is clear that, under eq. (18),  $Q(\mathbf{s}, \bar{\Theta})$  is minimized in terms of  $\bar{r}$  and  $\mathbf{e}_m$  and  $Q(\mathbf{s}, \bar{\Theta}) = J(\mathbf{s})$  is achieved when  $\bar{r} = (1/L) \sum_{m=1}^L r_m$  and  $\mathbf{e}_m = (\mathbf{s} - \mathbf{r}_m)/|\mathbf{s} - \mathbf{r}_m|$ . ■

### 3.3. Derivation of update rules

Unlike  $J(\mathbf{s})$ ,  $Q(\mathbf{s}, \Theta)$  has a simple quadratic form in terms of  $\mathbf{s}$ . Then,  $\partial Q/\partial \mathbf{s} = 0$  can be easily solved, which yields the update rule corresponding to eq. (5). While, the update rules corresponding to eq. (4) are already obtained in the proof of Lemma 3. They are summarized as follows:

$$\bar{r}^{(k+1)} = \frac{1}{L} \sum_{m=1}^L |\mathbf{s}^{(k)} - \mathbf{r}_m|, \quad (20)$$

$$\mathbf{e}_m^{(k+1)} = \frac{\mathbf{s}^{(k)} - \mathbf{r}_m}{|\mathbf{s}^{(k)} - \mathbf{r}_m|}, \quad (21)$$

$$\mathbf{s}^{(k+1)} = \frac{1}{L} \sum_{m=1}^L \left\{ \mathbf{r}_m + (\bar{r}^{(k+1)} + \bar{d}_m) \mathbf{e}_m^{(k+1)} \right\}, \quad (22)$$

where  $\bar{r}^{(k)}$ ,  $\mathbf{e}_m^{(k+1)}$ , and  $\mathbf{s}^{(k+1)}$  are the estimations of the average of source-sensor distances, the unit direction vector from the sensor  $m$  to the source, and the source position, respectively, at  $k$ th iteration.

Note that

$$\begin{aligned} \bar{d}_m &= \frac{1}{L} \sum_{n=1}^L d_{mn} \\ &= |\mathbf{s} - \mathbf{r}_m| - \frac{1}{L} \sum_{n=1}^L |\mathbf{s} - \mathbf{r}_n| + \frac{1}{L} \sum_{n=1}^L \varepsilon_{mn}, \end{aligned} \quad (23)$$

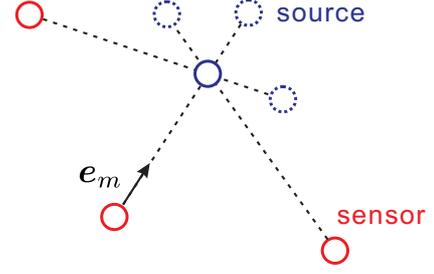
which means  $\bar{d}_m$  can be considered as an estimate of the difference between the source-sensor  $m$  distance and the average of the source-sensor distances. Thus, eq. (22) represents an averaging operation of  $L$  estimates of the source positions, which are obtained by updating the source-sensor  $m$  distance to  $(\bar{r}^{(k+1)} + \bar{d}_m)$  with the source direction fixed on each sensor. The eq. (18) requests that the estimate of the source-sensor  $m$  distance should be nonnegative, which can be satisfied in most cases.

Because of the resemblance of this iterative averaging operations to  $k$ -means clustering, we call it *r(range)-means localization*. The advantages of the r-means localization are: 1) it consists of simple iterative operations without matrix calculation, 2) the convergence is guaranteed, 3) it has no tuning parameters such as a step size in the steepest descent method.

Note that eq. (23) shows that RD-based localization problem is equivalent to range-based localization problem if the average of the source-sensor distances is correctly estimated. Actually, the r-means localization is consistent with Shi's iterative algorithm for range-based localization [11] except estimating the average of the source-sensor distances.

## 4. ACCELERATING CONVERGENCE

Although the update equations of r-means localization consists of simple averaging operations and they are fast calculated, the necessary iterations to convergence are not always a few. For more efficiency, applying acceleration method to r-means localization is considered here. The acceleration method is an algorithm to generate a faster-converged sequence



**Fig. 1.** The update of estimated source position in r-means localization: Through updating the average of source-sensor distances, the source position is updated by averaging  $L$  estimates, which are obtained by updating the source-sensor distance with the source direction fixed on each sensor.

$b_n$  from a given converged sequence  $a_n$ , which is well used in numerical calculation field. Since a sequence of iterative estimates by the auxiliary function method converges, it can be suitable with applying the acceleration method.

In Aitken acceleration [12], which is one of the most fundamental acceleration methods, an error model in  $\mathbf{s}^{(n)}$  converging to  $\mathbf{s}^*$ :

$$\mathbf{s}^{(n)} - \mathbf{s}^* = \lambda^n \mathbf{c} \quad (24)$$

is assumed where  $|\lambda| < 1$  and  $\mathbf{c}$  is a constant vector. Removing the unknown  $\mathbf{c}$  by subtracting  $\lambda$  times of  $\mathbf{s}^{(n-1)} - \mathbf{s}^* = \lambda^{n-1} \mathbf{c}$  from eq. (24), we have  $\mathbf{s}^* = \mathbf{s}^{(n)} + \alpha \Delta \mathbf{s}^{(n)}$ , where  $\alpha = \lambda/(1 - \lambda)$ ,  $\Delta \mathbf{s}^{(n)} = \mathbf{s}^{(n)} - \mathbf{s}^{(n-1)}$ . Although eq. (24) is not perfectly satisfied, a new sequence:

$$\mathbf{p}^{(n)} = \mathbf{s}^{(n)} + \alpha \Delta \mathbf{s}^{(n)} \quad (25)$$

can be considered to converge faster. If  $\mathbf{p}^{(n)} \simeq \mathbf{p}^{(n-1)}$  is assumed, the unknown parameter  $\alpha$  is determined as

$$\Delta \mathbf{s}^{(n)} \simeq -\alpha \Delta^2 \mathbf{s}^{(n)}, \quad (26)$$

where  $\Delta^2 \mathbf{s}^{(n)} = \Delta \mathbf{s}^{(n)} - \Delta \mathbf{s}^{(n-1)}$ .  $\alpha$  satisfying eq. (26) optimally in the least square sense is obtained by

$$\alpha = -\frac{\Delta \mathbf{s}^{(n)} \cdot \Delta^2 \mathbf{s}^{(n)}}{|\Delta^2 \mathbf{s}^{(n)}|^2}. \quad (27)$$

Furthermore, not only just obtaining the accelerated sequence  $\mathbf{p}^{(n)}$ , we can iteratively apply Aitken acceleration, which is called Steffensen acceleration [12]. We have to note that the non-increasing nature of the objective function is not always guaranteed in the accelerated sequence. Then, we obtain the accelerated sequence  $\tilde{\mathbf{s}}^{(n)}$  in the following way.

1. Set an initial value to  $\tilde{\mathbf{s}}^{(1)}$  and let iteration number  $k$  be set to 1.
2. Calculate  $\tilde{\mathbf{s}}^{(k+1)}$  from  $\tilde{\mathbf{s}}^{(k)}$  with using eq. (20), eq. (21) and eq. (22).
3. If  $k + 1$  is odd and  $k + 1 \geq 3$ , calculate  $\mathbf{p}^{(k+1)}$  from  $\tilde{\mathbf{s}}^{(k-1)}$ ,  $\tilde{\mathbf{s}}^{(k)}$  and  $\tilde{\mathbf{s}}^{(k+1)}$  with using eq. (25) and eq. (27). If  $J(\mathbf{p}^{(k+1)}) < J(\tilde{\mathbf{s}}^{(k+1)})$ , replace  $\tilde{\mathbf{s}}^{(k+1)}$  by  $\mathbf{p}^{(k+1)}$ .
4. Increment  $k$  and go to 2.

## 5. EXPERIMENTAL EVALUATION

We evaluated the validity of the r-means localization by simulative experiments. Sensors and sources were located randomly in two dimensional 5m×4m space. The range difference in each pair of sensors is assumed to be observed with Gaussian noise with zero mean and 0.01m standard deviation.

Fig. 2 shows examples of iterative estimates for five sources, where the number of sensors was five, the initial estimate of the source positions was set to be the center of gravity of sensors, and the number of iteration was 300. The loci is not always linear lines, but in all case, we can see that the estimates approach to the true source positions.

Then, in order to investigate the effect of the acceleration, the decreasing speed of the objective functions were compared among the r-means localization with and without applying Steffensen acceleration, and the steepest descent method with several step sizes. The results are shown in Fig. 3, where the number of sensors was five, and the average of the objective function for 100 source positions was calculated. The initial estimate of the source positions was again set to be the center of gravity of sensors. In the steepest descent method, the source position was iteratively estimated by  $\mathbf{s}^{(k+1)} = \mathbf{s}^{(k)} + a \cdot \partial J(\mathbf{s}^{(k)})/\mathbf{s}$  where  $a$  denotes step size.

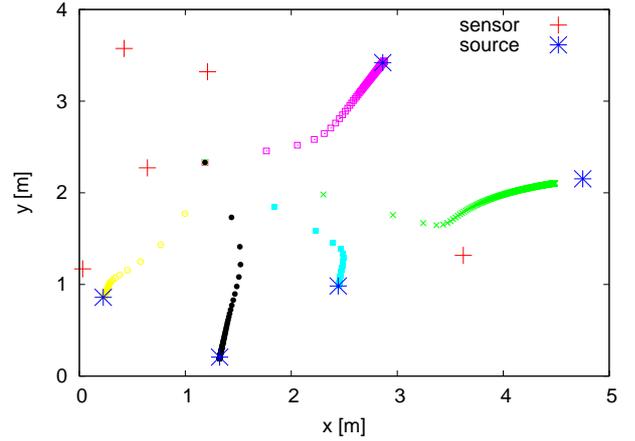
It is clear that in the steepest descent method the decreasing speed highly depends on the step size and it should also depend on number of sensors, noise level of RD observations, source sensor configurations, and so on. While, in the r-means localization, regardless of no tuning parameters, the objective function decreased as fast as the best one of the steepest descent method cases. Applying Steffensen acceleration much accelerated the convergence and 100 iterations seems to be sufficient. The calculation time for 100 iterations per single source with Steffensen acceleration was 0.1 ~ 0.3ms in a Laptop PC with 1GHz Pentium CPU.

## 6. CONCLUSION

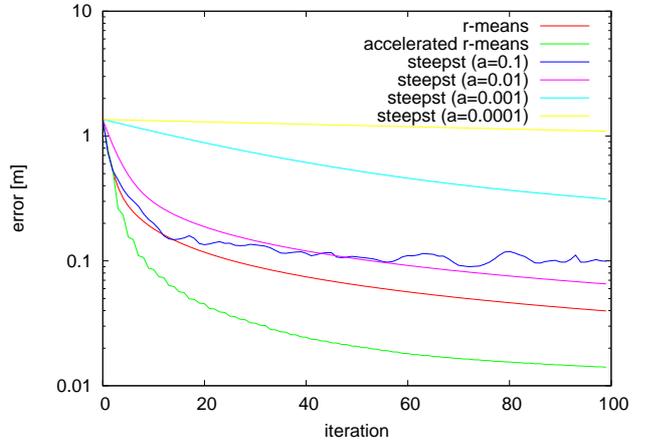
In this paper, we described a simple iterative algorithm, r-means localization, for RD-based localization, which consists of averaging operations. The acceleration of the convergence was also discussed. Applying this framework to microphone array and distributed sensor network is our current concern.

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**Fig. 2.** The loci of iteratively estimated source positions: sensor positions (red), true source positions (blue), iteratively estimated source positions (other colors)



**Fig. 3.** The comparison of the decreasing speed of the objective function: The vertical axis represents the averaged RD error per one source-sensor ( $\sqrt{J(\mathbf{s})}/(L(L-1))$ ). The horizontal axis represents the iteration number.

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